Work 功

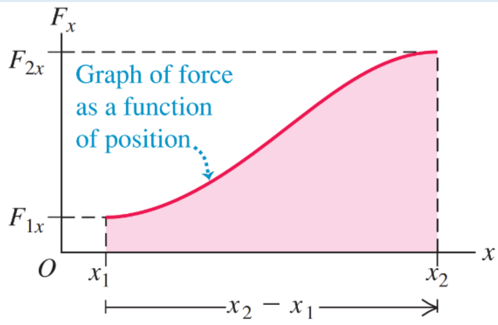
- $\vec{F}$ is a constant:

  $W = \vec{F} \cdot \vec{l} = Fl\cos\theta$

- If the object is traveling along a straight line:

  $$W = \int_{x1}^{x2} F_x dx$$

  Which is also equal to the area under the $F_x - x$ curve.



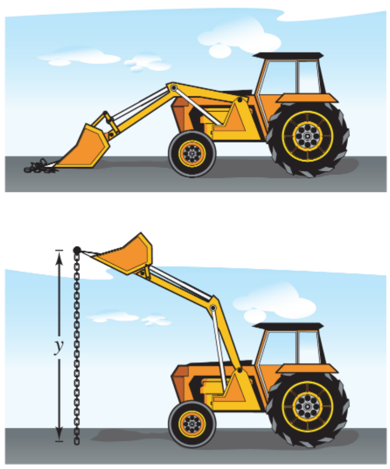
Some notice about the work:

- Work is a scalar, it can be positive or negative
- If more than 1 force acts on the object, the total work equals the sum of the work done by each individual force

$$W = \int_l \vec{F}_{net} \cdot d\vec{l} = \int_l (\vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n) \cdot d\vec{l}$$

$$= \int_l \vec{F}_1 \cdot d\vec{l} + \int_l \vec{F}_2 \cdot d\vec{l} + \cdots \int_l \vec{F}_n \cdot d\vec{l} = W_1 + W_2 + \cdots W_n$$

Example: A 20-foot chain weighing 5 pounds per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 20 feet so that it is fully extended, as shown in the right figure.



Imagine that the chain is divided into small sections, each of length dy. Then the weight of each section is the increment of force

$$dF = (weight) = \left(\frac{5\ pounds}{foot}\right) length = 5dy$$

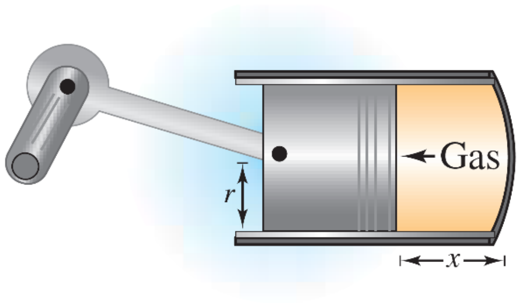
Because a typical section (initially on the ground) is raised to a height of y, the increment of work is

$$dW = (force\ increment)(distance) = dF(y) = 5ydy$$

Because y ranges from 0 to 20, the total work is

$$W = \int_0^{20} 5ydy = \left(\frac{5y^2}{2}\right)_0^{20} = \frac{5(400)}{2} = 1000 \quad (foot \cdot pound)$$

Example: A quantity of gas with an initial volume of 1 cubic foot and a pressure of 500 pounds per square foot expands to a volume of 2 cubic feet. Assume that the pressure is inversely proportional to the volume. Find the work down by the gas.



Pressure-volume: $p = k/V \Rightarrow k = pV = 500 \cdot 1 = 500$

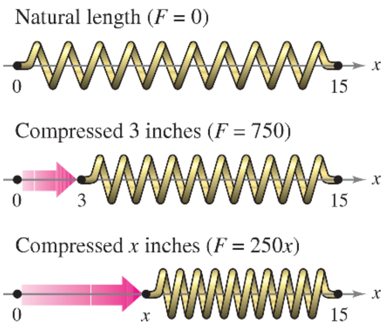
Volume-displacement: $V = sx \Rightarrow dV = sdx$

where s is the area of the cylinder

$Fdx = psdx = pdV = (k/V)dV$

$$W = \int_{V0}^{V1} \frac{k}{v} dV = \int_{1}^{2} \frac{500}{v} dV = (500 \ln|V|)_1^2 \approx 346.6 \ (foot \cdot pound)$$

Exercise 12: A Force of 750 pounds compresses a spring 3 inches from its natural length of 15 inches. Find the work down in compressing the spring an additional 3 inches.



By Hooke's law:

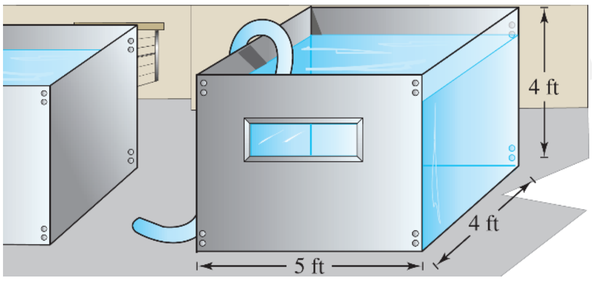
$F(x) = kx \ \Rightarrow$

$k = F/x = 750/3 = 250 \ \ (pound/inch)$

The spring will be compressed from 3 inches to 6 inches

$$W = \int_{x1}^{x2} F_x dx = \int_{3}^{6} (250x) dx = (125x^2)_3^6 = 3375 \ (pound \cdot inch)$$

Exercise 13: A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water. The water's density is ρ=62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty half of the tank?



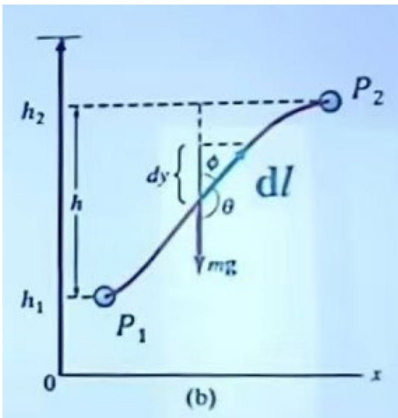
Water's mass: $m = \rho V,\ V = sx,\ s = 4 \cdot 5$ (square feet)

$$\text{Work: } W = \int_0^2 F dx = \int_0^2 mg dx = \int_0^2 \rho V dx = \int_0^2 \rho s x dx = \left(\rho s \frac{x^2}{2}\right)_0^2$$

$$= 62.4 \cdot 4 \cdot 5 \cdot \frac{4}{2} = 2496 \ (feet \cdot pound)$$

Work done by gravitational force near the surface of the Earth

For an object of mass m moves from p_1 to p_2 along path L.



$$\vec{F} = -mg\vec{j}$$

$$d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$dW = \vec{F} \cdot d\vec{l} = -mgdy$$

$$\Rightarrow W = \int \vec{F} \cdot d\vec{l} = \int_{h_1}^{h_2} -mgdy$$

$$= -(mgh_2 - mgh_1)$$

Conservative force and potential energy 保守力和势能

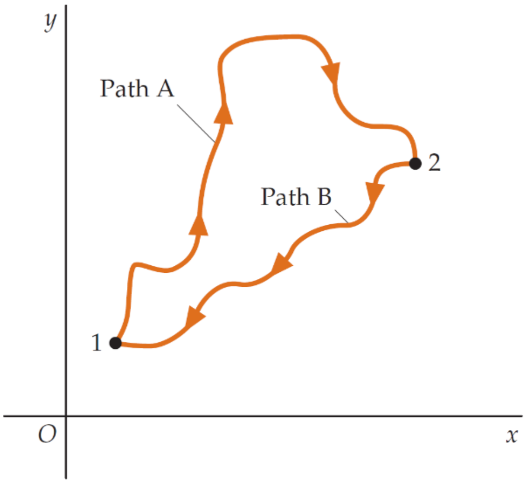
For a conservative force:

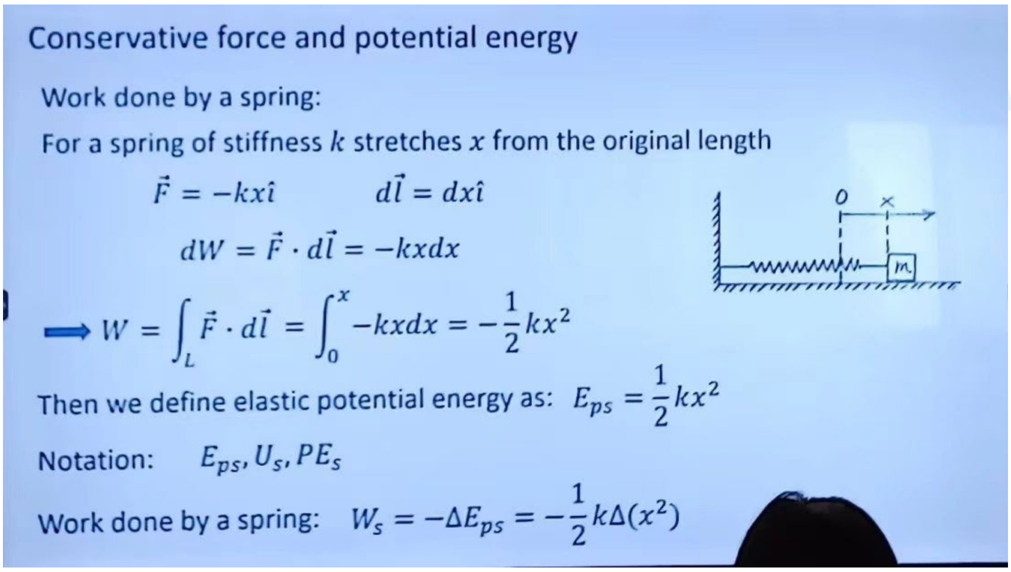
$$W = \int_L \vec{F} \cdot d\vec{l} = -\left(E_p(B) - E_p(A)\right)$$

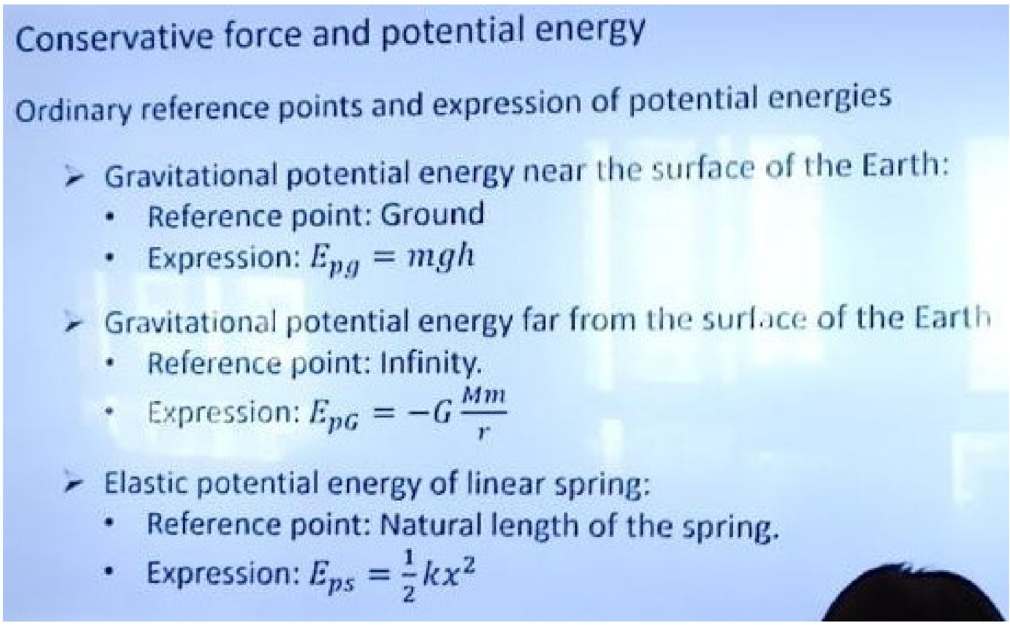
- E_p is a function only depend on the position of the object?
- E_p has the unit of energy.

We denote the function E_p as the potential energy of the corresponding conservative force F.

For each conservative force, there is always a potential energy associated with the force.







Kinetic-Energy and the Work-Energy Theorem 功能定理

Kinetic Energy

$$E_k = \frac{1}{2}mv^2$$

Work-Energy Theorem

$$W_{tot} = E_{k2} - E_{k1} = \Delta E_k$$

Work-energy theorem (Theorem of kinetic energy)

What does the work change the status of the object?

$$dW = \vec{F} \cdot d\vec{l}$$

Recap: Newton's second law:

$$\vec{F} = m\vec{a} \Longrightarrow dW = \vec{F} \cdot d\vec{l} = m\frac{d\vec{v}}{dt} \cdot d\vec{l} = m\vec{v} \cdot d\vec{v}$$

Related math equation: $d\left(\vec{f} \cdot \vec{g}\right) = d\vec{f} \cdot \vec{g} + \vec{f} \cdot d\vec{g}$

$$d(\vec{v} \cdot \vec{v}) = \vec{v} \cdot d\vec{v} + \vec{v} \cdot d\vec{v} = 2\vec{v} \cdot d\vec{v}$$

$$d(\vec{v} \cdot \vec{v}) = d(v^2) = 2vdv$$

$$\Longrightarrow m\vec{v} \cdot d\vec{v} = mvdv$$

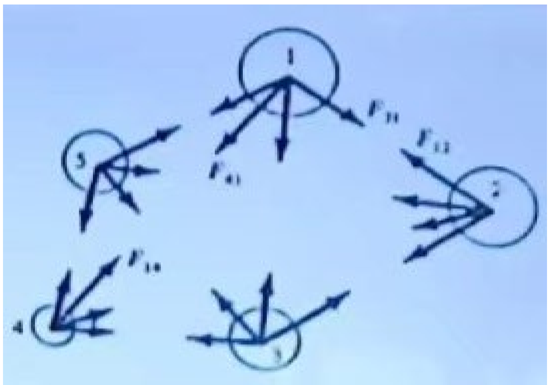
$$\boxed{dW = mvdv}$$

Work-energy theorem (Theorem of kinetic energy)

If a particle moves in such a way that its position x is described as a function of time t by $x = t^{3/2}$, then its kinetic energy is proportional to

(A) t^2
(B) $t^{3/2}$
(C) t
(D) $t^{1/2}$
(E) t^0 (i.e., kinetic energy is constant)

Work-Energy Theorem for a Particle System



Considering a system consists of n particles: work done on the i-th particle is W_i

$$W_i = \Delta E_{ki}$$

For all of the particles:

$$\sum W_i = \sum \Delta E_{ki}$$

$$\Rightarrow \sum W_i = W_{t-int} + W_{t-ext}$$

Work done by internal force

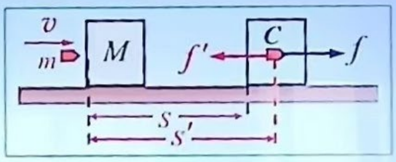
Work done by external force

Work-Energy Theorem for a particle system:

Work done by internal force:

$$W_f = fs \qquad W_f' = -f's'$$

$$\sum W_i = W_f + W_f' = -f(s' - s) < 0$$

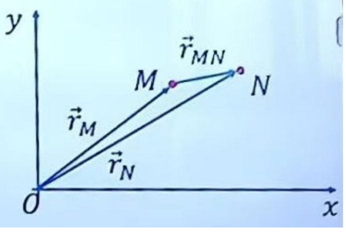


Total work done by internal force in general case:

$$dW = \vec{F}_{MN} d\vec{r}_N + \vec{F}_{NM} d\vec{r}_M$$

$$= \vec{F}_{MN} \, (d\vec{r}_N - d\vec{r}_M)$$

$$= \vec{F}_{MN} \, d(\vec{r}_N - \vec{r}_M)$$



Conservation of Energy 能量守恒定理

Conservation of energy

Review the theorem of kinetic energy

$$W_{net} = \Delta E_k$$

If we sperate the work as work done by conservative force plus the work done by nonconservative force.

$$W_{net} = W_{nc} + W_c = \Delta E_k$$

$$W_{nc} = \Delta E_k + \Delta E_p$$

The work done by the nonconservative force act on an object equals to the change in kinetic and potential energies.

Conservative force and potential energy

Relation between conservative force and potential energy

$$W = \int_L \vec{F} \cdot d\vec{l} = -(E_p(B) - E_p(A)) \Rightarrow W = -\Delta E_p$$

Work done by the conservative force act on an object equals to the negative value of change potential energy.

If the force is only along the x axis:

$$dW = F_x \cdot dx = -dE_p \Rightarrow \boxed{F_x = -\frac{dE_p}{dx}}$$

Conservation of energy

Other forms of energy:

Electric energy, nuclear energy, thermal energy, and the chemical energy, *etc*.

- Energy can be transformed from one form to another.
- Work is done when energy is transferred from one object to another.

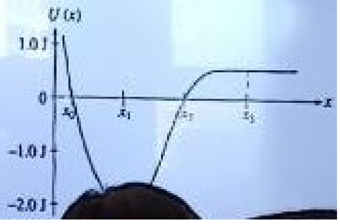
Law of conservation of energy:

The total energy is neither increased nor decreased in any process. Energy can be transformed from one form to another, and transferred from one object to another, but the total amount remains constant.

Conservation of energy

A conservative force has the potential energy function $U(x)$, shown by the graph. A particle moving in one dimension under the influence of this force has kinetic energy 1.0 joule when it is at position x_1 Which of the following is a correct statement about the motion of the particle?

(A) It oscillates with maximum position x_2 and minimum position x_0
(B) It moves to the right of x_3 and does not return.
(C) It moves to the left of x_0 and does not return.
(D) It comes to rest at either x_0 or x_2.
(E) It cannot reach either x_0 or x_2.



Conservative force and potential energy

A uniform chain of mass M and length l hangs from a hook in the ceiling. The bottom link is now raised vertically and hung on the hook as shown on the right.

a. Determine the increase in gravitational potential energy of the chain by considering the change in position of the center of mass of the chain.

Conservative force and potential energy

A uniform chain of mass M and length l hangs from a hook in the ceiling. The bottom link is now raised vertically and hung on the hook as shown on the right.

b. Write an equation for the upward external force $F(y)$ required to lift the chain slowly as a function of the vertical distance y.

Center of Mass Reference Frame 质心参考系(Optional)

Center of mass reference frame

Konig's theorem: The kinetic energy of a system of particles is the sum of the kinetic energy associated to the movement of the center of mass and the kinetic energy associated to the movement of the particles relative to the center of mass.

$$E_k = \frac{1}{2}Mv_{cm}^2 + \sum_{i=1}^{n}\frac{1}{2}m_i v_i^{c2}$$

Konig's theorem is true no matter the center of mass reference frame is inertial no not.

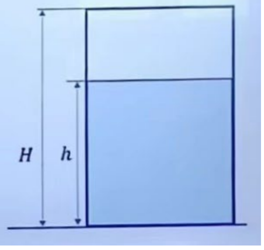
Center of mass

A uniform cup of mass M, height H, and cross sectional area A is placed on a horizontal surface. Liquid of density ρ is filled in the cup and the height of the liquid in the cup is h. Ignore the thickness and mass of the bottom of the cup.
(a) Determine the location of the center of mass of the system.
(b) Determine the expression of h in terms of M, H, A, ρ and proper constant when the location of the center of mass is lowest.

$$y_{sys} = \frac{HM + \rho A h^2}{2(M + \rho A h)}$$

When the center of mass is lowest:

$$\frac{dy_{sys}}{dh} = 0 \Rightarrow h = \frac{\sqrt{M^2 + \rho AHM} - M}{\rho A}$$



Center of mass

Try to calculate the position of center of mass for a rod with length L and linear density $\lambda = \lambda_0 l$ where λ_0 is a proper constant.

x coordinate of center of mass: $x_{cm} = \frac{\int x dm}{\int dm}$

$$\int dm = \int_0^L \lambda_0 l \, dl = \frac{\lambda_0 L^2}{2} \qquad \int x dm = \int_0^L \lambda_0 l^2 \, dl = \frac{\lambda_0 L^3}{3}$$

$$\Rightarrow x_{cm} = \frac{2L}{3}$$

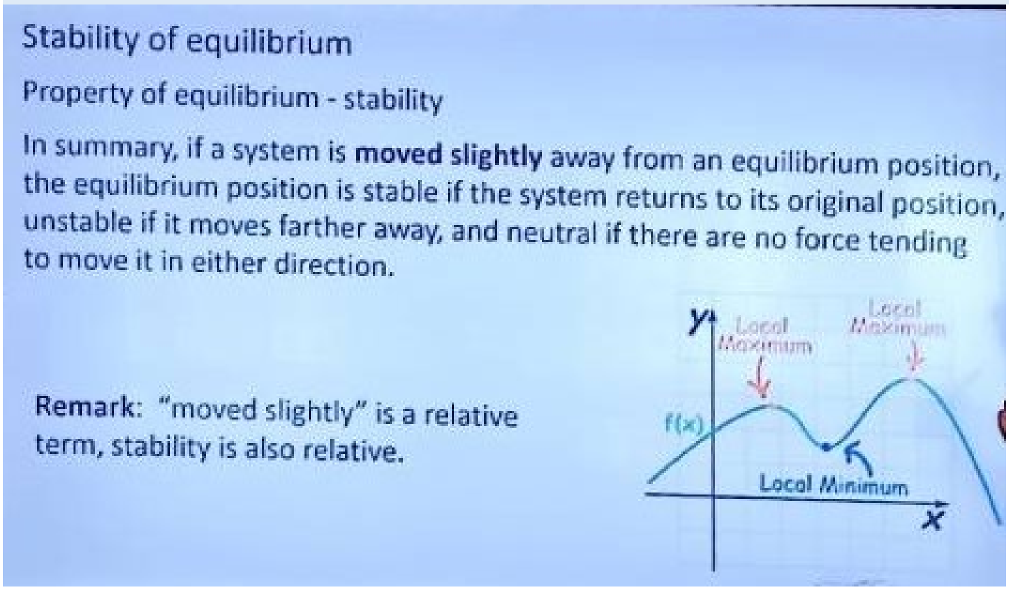
Position is 2/3 to the end with less linear density

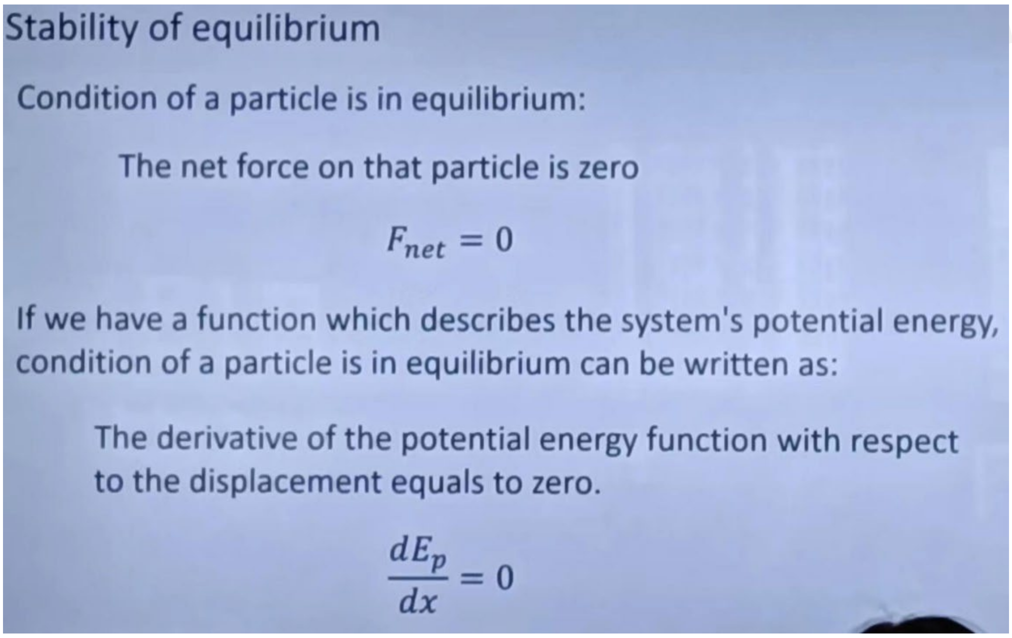
Center of mass

A balloon of mass M is floating motionless in the air. A person of mass less than M is on a rope ladder hanging from the balloon. The person begins to climb the ladder at a uniform speed v relative to the ground. How does the balloon move relative to the ground?

(A) Up with speed v
(B) Up with a speed less than v
(C) Down with speed v
(D) Down with a speed less than v
(E) The balloon does not move.

Stability of Equilibrium 平衡的稳定性(Optional)

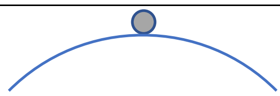




Property of equilibrium – stability

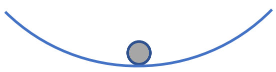
Neutral equilibrium : no force arises from a small displacement of the object from equilibrium that tends to drag it either backward, or away from, its original position.

Unstable equilibrium: the force that arise from a small displacement of the object from equilibrium tend to push the object even farther away from its equilibrium position



$$\frac{\mathrm{d}^2 E_p}{dx^2} < 0 \Rightarrow \text{local maximum of the potential energy}$$

Stable equilibrium: the force that arise from a small displacement of the object from equilibrium tend to dray the object even farther backward toward its equilibrium position

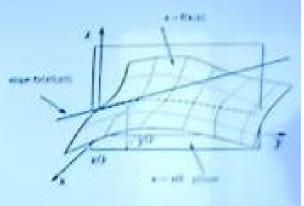


$$\frac{\mathrm{d}^2 E_p}{dx^2} < 0 \Rightarrow \text{local minimum of the potential energy}$$

Stability of equilibrium (Optional)

Directional derivative and gradient:

For a scalar function $z = f(x, y)$, directional derivative along the direction $\vec{u} = (\cos\theta, \sin\theta)$ is



$$\frac{\partial f}{\partial u} = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

For a scalar function $z = f(x, y)$, gradient of f is defined as

$$\text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$ ⟹ Gradient is a vector

Relation between directional derivative and gradient: $\frac{\partial f}{\partial u} = \nabla f \cdot \vec{u}$

The direction of maximum increase of f is given by ∇f

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